# GPS Composite Clock Analysis

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*Abstract*— **The GPS Composite Clock defines GPS Time, the timescale used today in GPS operations. GPS Time is illuminated by examination of its role in the complete estimation and control problem relative to UTC/TAI. Simulated GPS clock phase and frequency deviations, and simulated GPS pseudo-range measurements, are used to understand GPS Time in terms of Kalman filter estimation errors.**

CONTENTS



## I. INTRODUCTION

GPS Time is *created* by processing GPS pseudo-range measurements with the operational GPS Kalman filter. Brown[1] refers to the object created by the Kalman filter as the GPS Composite Clock, and to GPS Time as the Implicit Ensemble Mean phase of the GPS Composite Clock. The fundamental goal by the USAF and the USNO is to control GPS Time to within a specified bound of UTC/TAI. I present here a quantitative analysis of the GPS Composite Clock, derived from detailed simulations and associated graphics. GPS clock diffusion coefficient values used here were derived from Allan Deviation graphs presented by Oaks[8] et. al. in 1998. I refer to them as "realistic", and in the sequel I claim "realistic" results from their use. Fig. 1 presents their diffusion coefficient values and my derivation of associated Allan Deviation lines.

My interest in the GPS Composite Clock derives from my interest in performing real-time orbit determination[1] for GPS NAVSTAR spacecraft from ground receiver pseudo-range measurements. The estimation of NAVSTAR orbits would be incomplete without the simultaneous estimation of GPS clock parameters. I use simulated GPS clock phase and frequency deviations, and simulated GPS pseudo-range measurements, to study Kalman filter estimation errors.

This paper was prepared for TimeNav'07[2]. I am indebted to Charles Greenhall (JPL) for encouragement and help in this work.

## II. THE COMPLETE ESTIMATION AND CONTROL PROBLEM

The USNO operates two UTC/TAI master clocks, each of which provides access to an estimate of UTC/TAI in real-time (1 pps). One of these clocks is maintained at the USNO, and the other is maintained at Shriever Air Force Base in Colorado Springs. This enables the USNO to compare UTC/TAI to the phase of each GPS orbital NAVSTAR clock via GPS pseudo-range measurements and two-way time transfer, by embedding a UTC/TAI master clock in a USNO GPS ground receiver. Each GPS clock is a member of (internal to) the GPS ensemble of clocks, but the USNO master clock is external to the GPS ensemble of clocks. Because of this, the difference between UTC/TAI and the phase of each NAVSTAR GPS clock can be (and is) estimated and quantified. The RMS (Root Mean Square) on these differences quantifies the difference between UTC/TAI and GPS Time. Inspection of the differences between UTC/TAI and the phase of each NAVSTAR GPS clock enables the USNO to identify GPS clocks that require particular frequency-rate control corrections. Use of this knowledge enables the USAF to adjust frequency rates of selected GPS clocks. Currently the USAF uses an automated

[1]James R Wright is the architect of ODTK (Orbit Determination Tool Kit), a commercial software product offered by Analytical Graphics, Inc. (AGI).



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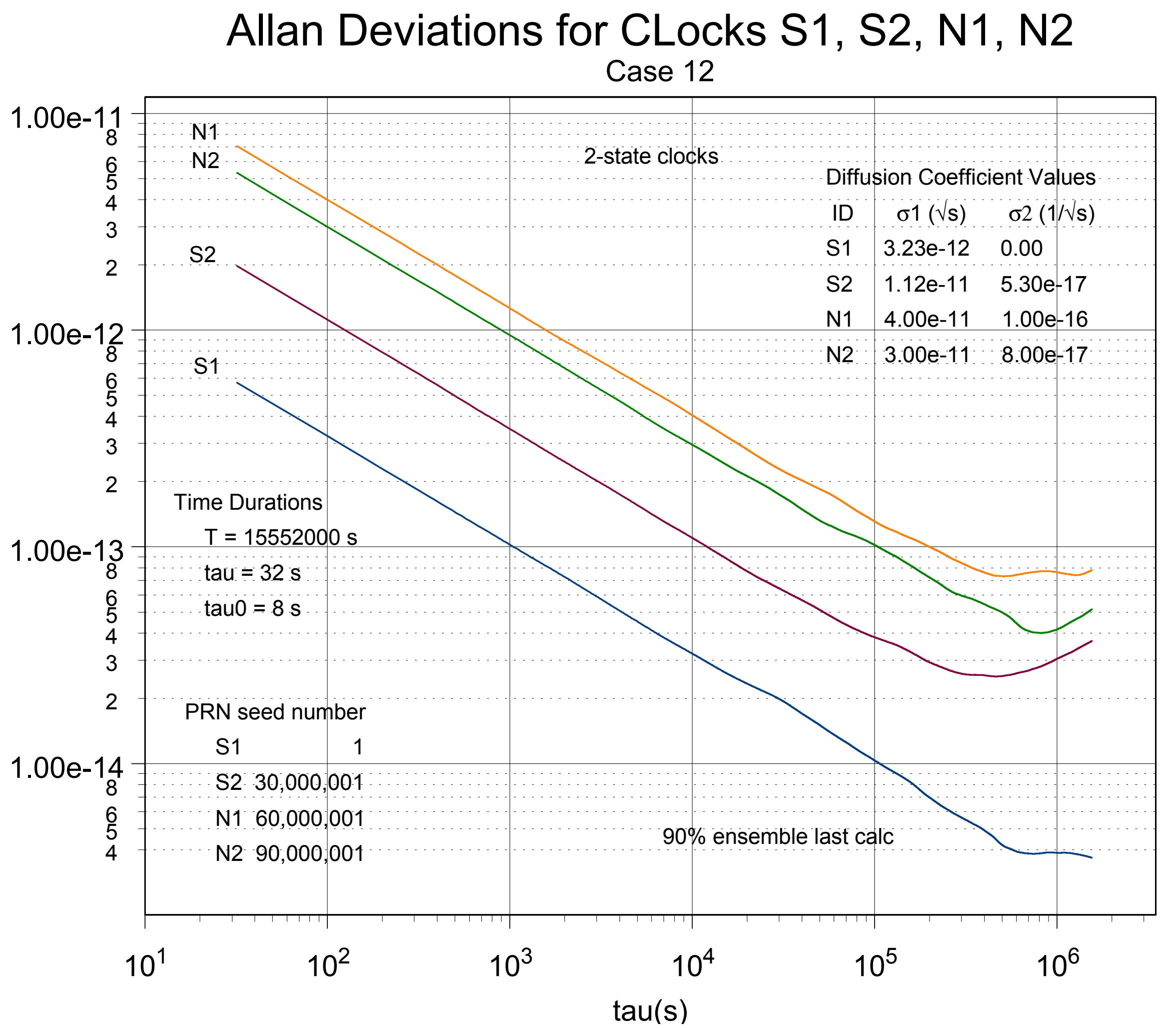


Fig. 1. Allan Deviation Lines for S1, S2, N1, and N2

bang-bang controller[3] on frequency-rate.

## III. Clock Physics

The most significant clock physics are understood in terms of Wiener processes and their integrals. Clock physics are characterized by particular values of clock-dependent diffusion coefficients, and are conveniently studied with aid of a relevant clock model that relates diffusion coefficient values to their underlying Wiener processes. For my presentation here I have selected "The Clock Model and Its Relationship with the Allan and Related Variances” presented as an IEEE paper by Zucca and Tavella[14] in 2005. Except for FM flicker noise, this model captures the most significant physics for all GPS clocks. I simulate and validate GPS pseudo-range measurements using simulated phase deviations and simulated frequency deviations, according to Zucca and Tavella.

[3]According to Bill Feess, an improvement in control can be achieved by replacing the existing "bang-bang controller" with a "proportional controller".

## IV. Allan Variance and PPN Relations

### A. Allan Coefficients vs Diffusion Coefficients

Denote $\tau$ as clock averaging time, $\sigma_y^2(\tau)$ as Allan variance, $a_0$ as Allan's FMWN coefficient, $a_{-2}$ as Allan's FMRW coefficient, $\sigma_1$ as the FMWN diffusion coefficient, and $\sigma_2$ as the FMRW diffusion coefficient. Then:

$$\sigma_y^2(\tau) = a_0\tau^{-1} + a_{-2}\tau = \sigma_1^2\tau^{-1} + \frac{1}{3}\sigma_2^2\tau \qquad (1)$$

where:

$$\sigma_1 = \sqrt{a_0} \qquad (2)$$

$$\sigma_2 = \sqrt{3a_{-2}} \qquad (3)$$

### B. Proportionate Process Noise (PPN)

Let $\alpha$ denote a variable $\alpha \in \{1, 2, 3, \ldots, N\}$ to identify each GPS clock in an ensemble of $N$ clocks. For each clock

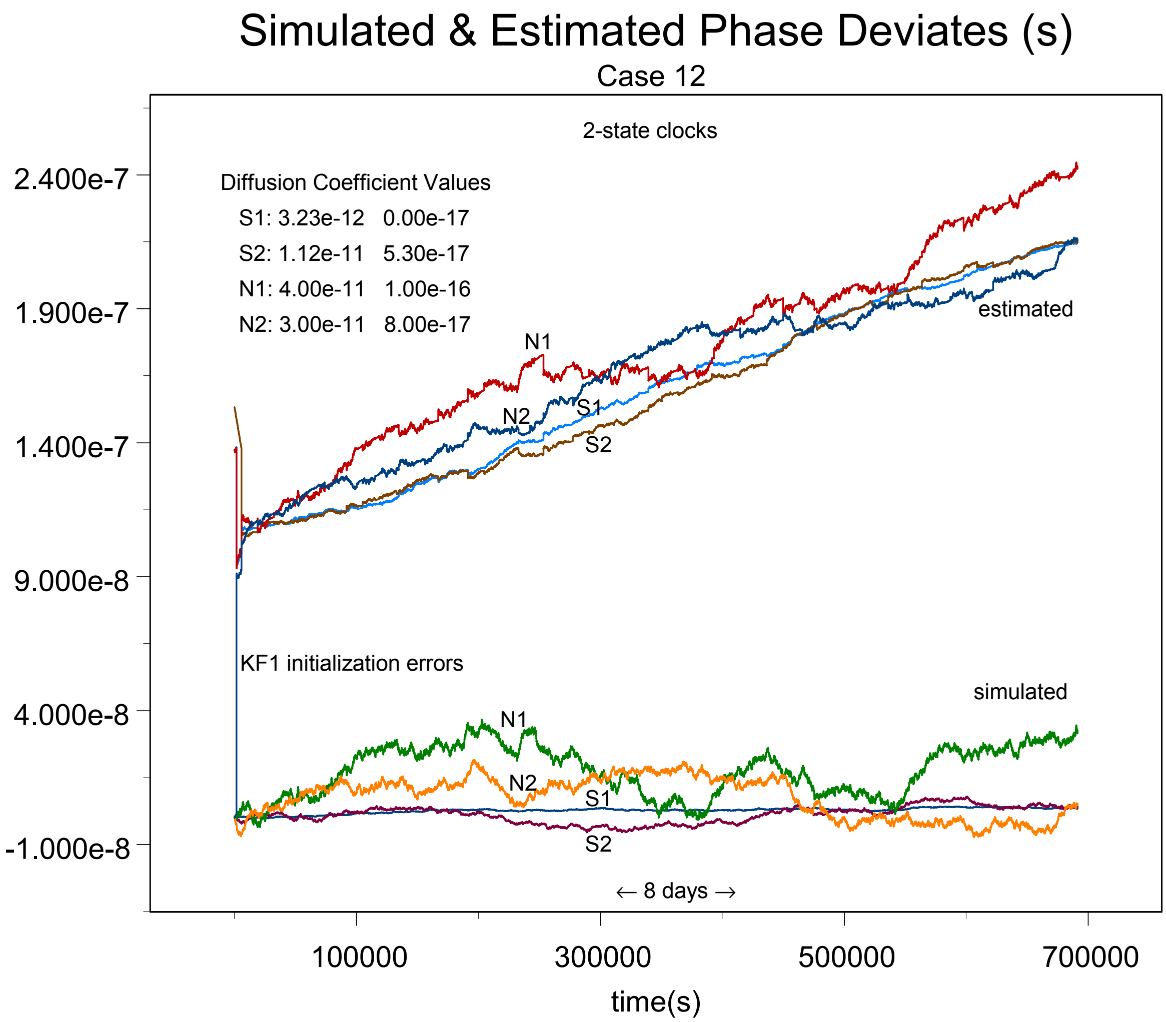


Fig. 2. Simulated and Estimated Phase Deviations for Four 2-State Clocks

$\alpha$ define the ratio $S_\alpha$ between diffusion coefficients $\sigma_{1\alpha}$ and $\sigma_{2\alpha}$:

$$S_\alpha = \frac{\sigma_{2\alpha}}{\sigma_{1\alpha}}, \qquad \sigma_{1\alpha} > 0 \tag{4}$$

Then PPN is defined when, for each GPS clock $\alpha$ and each associated ratio $S_\alpha$, we have:

$$S_1 = S_2 = S_3 = \cdots = S_N \tag{5}$$

*C. Case 12*

The calculation of $S_\alpha = \sigma_{2\alpha}/\sigma_{1\alpha}$, $\alpha \in \{1,2,3,4\}$, according to the diffusion coefficient values presented in Fig. 1 shows that PPN is *not* satisfied for Case 12:

$$\frac{\sigma_{2S1}}{\sigma_{1S1}} = 0.00 \times 10^{-6}\ \mathrm{s}^{-1}$$

$$\frac{\sigma_{2S2}}{\sigma_{1S2}} = 4.73 \times 10^{-6}\ \mathrm{s}^{-1}$$

$$\frac{\sigma_{2N1}}{\sigma_{1N1}} = 2.50 \times 10^{-6}\ \mathrm{s}^{-1}$$

$$\frac{\sigma_{2N2}}{\sigma_{1N2}} = 2.67 \times 10^{-6}\ \mathrm{s}^{-1}$$

## V. Kalman Filters KF1 and KF2

I have simulated GPS pseudo-range measurements for two GPS ground station clocks S1 and S2, and for two GPS NAVSTAR clocks N1 and N2. Here I set simulated measurement time granularity to 30s for the set of all visible link intervals. Visible and non-visible intervals are clearly evident in the blue line of Fig. 4. I set the scalar root-variance $\sqrt{R}$ for both measurement simulations and Kalman filter KF1 to $\sqrt{R} = 1$ cm. Typically $\sqrt{R} \sim 1$ m for GPS pseudo-range, but when carrier phase measurements are processed simultaneously with pseudo-range, the root-variance is reduced by two orders of magnitude. So the use of $\sqrt{R} = 1$ cm enables me to quantify lower performance bounds for the simultaneous processing of both measurement types.

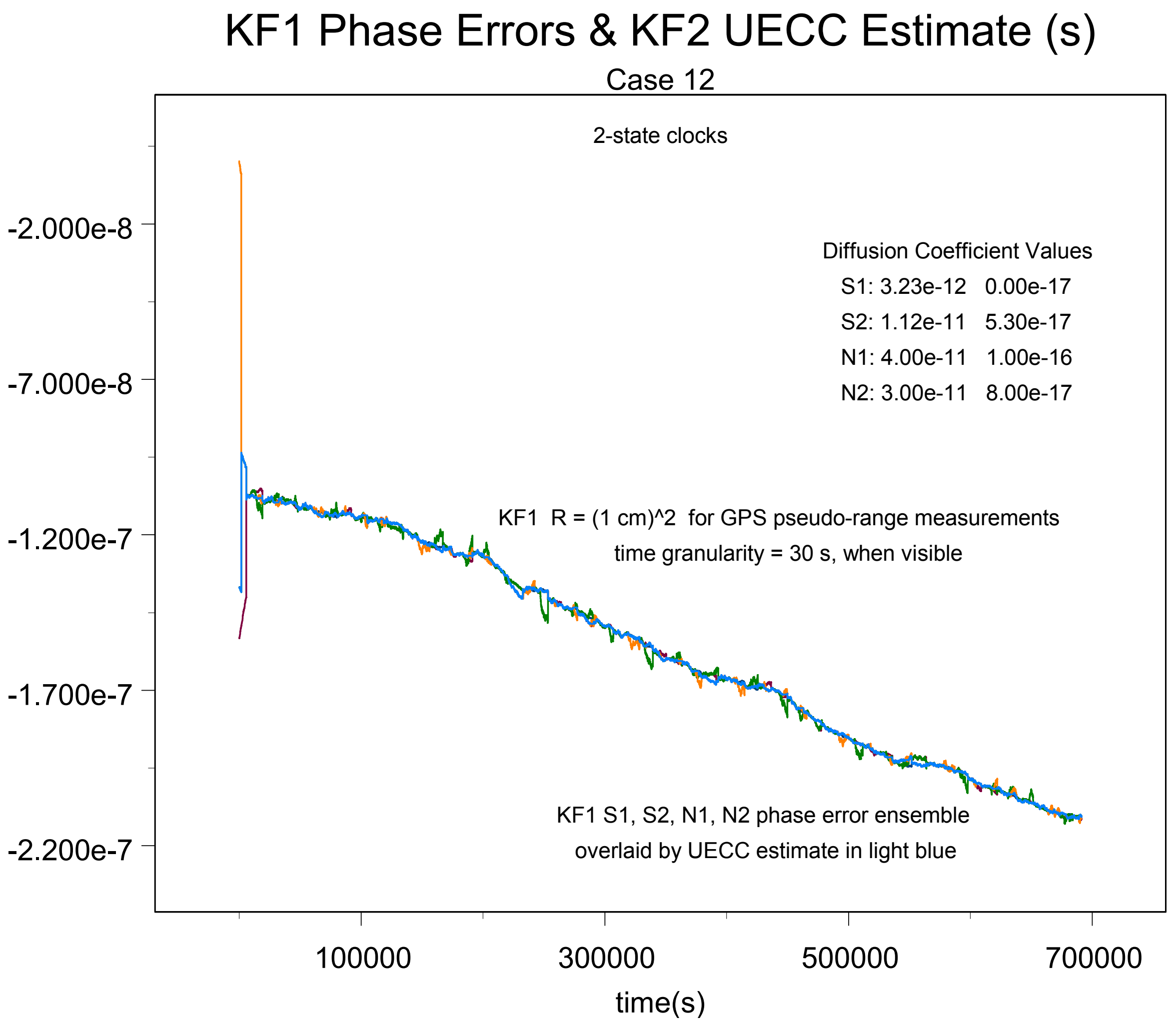


Fig. 3. KF1 Phase Errors and KF2 UECC Estimate

### *A. Create GPS Clock Ensemble*

Typically, one processes measurements with a Kalman filter to derive sequential estimates of a multidimensional *observable* state. Instead, here I imitate the GPS operational procedure and process simulated GPS pseudo-range measurements with KF1 to *create* a sequence of unobservable multi-dimensional clock state estimates. Clock state components are unobservable from GPS pseudo-range measurements. See Fig. 2 for an example of an ensemble of estimated unobservable clock phase deviation state components created by KF1.

*1) Sherman's Theorem:* GPS Time, the unobservable GPS clock ensemble mean phase, is created by the use of Sherman's Theorem[7][13] in the USAF Kalman filter measurement update algorithm on GPS range measurements. Satisfaction of Sherman's Theorem guarantees that the mean-squared state estimate error on each *observable* state estimate component is minimized. But the mean-squared state estimate error on each *unobservable* state estimate component is not reduced. Thus the unobservable clock phase deviation state estimate component common to every GPS clock is isolated by application of Sherman's Theorem. An ensemble of unobservable state estimate components is thus created by Sherman's Theorem – see Fig. 3 for an example.

### *B. Initial Condition Errors*

A significant result emerges due to the modeling of Kalman filter (KF1) initial condition errors in phase and frequency. Initial estimated clock phase deviations are significantly displaced by the KF1 initial condition errors in phase. As time evolves estimated clock phase deviation magnitudes diverge continuously and increasingly when referred to true (simulated) phase deviations, and this is due to filter initial condition errors in frequency. See Fig. 2 for an example.

### *C. Partition of KF1 Estimation Errors*

Subtract estimated clock deviations from simulated (true) clock deviations to define and quantify Kalman filter (KF1) estimation errors. Adopt Brown's additive partition of KF1 estimation errors into two components. I refer to the first

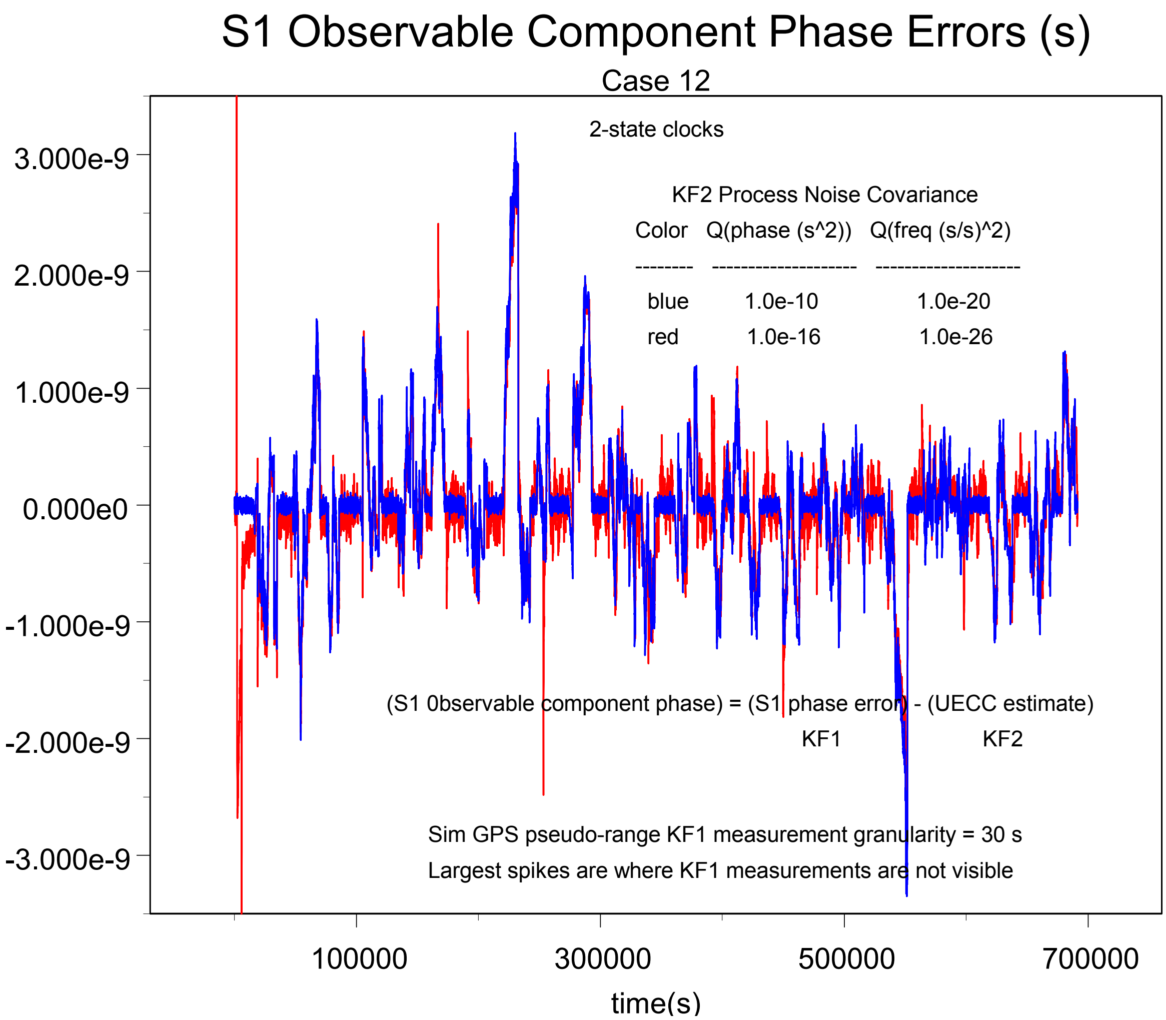


Fig. 4. S1 Observable Component Phase Error

component as the **U**nobservable **E**rror **C**ommon to each **C**lock (UECC), and to the second component as the **O**bservable[4] **E**rror **I**ndependent for each **C**lock (OEIC). Initially the variances on the UECC and OEIC are identical. On processing the first GPS pseudo-range measurements with KF1 the variances on both fall quickly. But with continued measurement processing the variances on the UECC increase without bound while the variances on the OEIC appoach zero asymptotically.

For simulated GPS pseudo-range data I create an optimal sequential estimate of the UECC by application of a second Kalman filter KF2 to pseudo-measurements defined by the phase components of KF1 estimation errors.

Since there is no physical process noise[5] on the UECC, an estimate of the UECC can also be achieved using a batch least squares estimation algorithm on the phase components of KF1 estimation errors – demonstrated previously by Greenhall[4].

[4]Observability is meaningful here only when processing *simulated* GPS pseudo-range data.

[5]I apply sufficient process noise covariance for KF2 to mask the effects of double-precision computer word truncation. Without this, KF2 does diverge.

### D. *Unobservable Error Common to each Clock*

There are at least four techniques to estimate the UECC when simulating GPS pseudo-range data. First, one could take the sample mean of KF1 estimation errors across the clock ensemble at each time and form a sample variance about the mean; this would yield a sequential sampling procedure, but where each mean and variance is sequentially unconnected. Second, one can employ Ken Brown's Implicit Ensemble Mean (IEM) and covariance; this is a batch procedure requiring an inversion of the KF1 covariance matrix followed by a second matrix inversion of the modified covariance matrix inverse; this is not a sequential procedure. Third, one can adopt the new procedure by Charles Greenhall[4] wherein KF1 phase estimation errors are treated as pseudo-measurements, and are processed by a batch least squares estimator to obtain optimal batch estimates and covariance matrices for the UECC. Fourth, one can treat the KF1 phase estimation errors as pseudo-measurements, invoke a second Kalman filter (KF2), and process these phase pseudo-measurements with KF2 to obtain optimal sequential estimates and variances for the UECC. I have been successful with this approach. Fig. 3 presents an ensemble of "realistic"[6] KF1 phase estimation errors, overlaid with "realistic" KF2 sequential estimates of UECC in phase.

### *E. Observable Error Independent for each Clock*

At each applicable time subtract the estimate of the UECC from the KF1 phase deviation estimate, for each particular GPS clock, to estimate the OEIC for that clock. Fig. 4 presents a graph of two cases of the OEIC for ground station clock S1. For the blue line of Fig. 4, intervals of link visibility and KF1 range measurement processing are clearly distinguished from propagation intervals with no measurements. During measurement processing, the observable component of KF1 estimation error is contained within an envelope of a few parts of a nano-second.

Calculation of the sequential covariance for the OEIC requires a matrix value for the cross-covariance between the KF1 phase deviation estimation error and the UECC estimation error at each time. I have not yet been able to calculate this cross-covariance.

## VI. Identify Non-Clock Modeling Errors

My interest in the GPS NAVSTAR (SV) orbit determination problem, combined with that of the clock parameter estimation problem, has enabled the identification of a useful diagnostic tool: Given realistic values for diffusion coefficients for each of the real GPS clocks, then quantitative upper bounds can be calculated on OEIC magnitudes. These calculations require the use of a rigorous *simulator*. Existence of significant cross-correlations between GPS clock phase errors and other non-clock GPS estimation modeling errors enables significant aliasing into GPS clock phase estimates during operation of KF1 on *real* data. But given rigorous quantitative upper bounds on OEIC magnitudes, then significant violation of these bounds when processing *real* GPS pseudo-range and carrier phase data identifies non-clock modeling errors related to the GPS estimation model. Modeling error candidates here include NAVSTAR orbit force modeling errors, ground antenna modeling errors (multipath), and tropospheric modeling errors. NAVSTAR orbit force modeling errors include those of solar photon pressure, albedo, thermal dump, and propellant outgassing. The accuracy of this diagnostic tool depends on the use of realistic clock diffusion coefficient values and a rigorous clock model simulation capability.

## VII. Observable Clocks

In the original 21-page version of my paper, I reported on KF1 validation results where S1 was specified as a TAI/UTC clock, external to the GPS clock ensemble consisting of S2, N1, and N2. This brought observablity to all GPS clock states from GPS pseudo-range measurements, drove clocks S2, N1, and N2 immediately to the TAI/UTC timescale, and enabled a clean validation of my filter implementation. Also it raised the question: Why not do the same thing for the real GPS clock ensemble? Discussions with Ed Powers (USNO) and Bill Feess (Aerospace Corporation) reveal that this approach was tried and discarded after the difficulty in recovery from an uplink hardware failure was blamed on the use of a single TAI/UTC Master Clock.

### *A. Recommendation for GALILEO Managers*

Embed at least one replaceable GALILEO range and carrier-phase ground receiver with at least one replaceable real-time TAI/UTC clock into your ensemble of GALILEO clocks, and commit resources to derive a very high level of system reliability. This will enable driving the mean phase of the entire GALILEO clock system to TAI/UTC in near-real-time, and will simplify immensely the understanding, implementation, estimation, validation, and control of the desired relation between the phase of each GALILEO clock and TAI/UTC. Do *not* create a timescale independent of TAI/UTC.

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[6] By "realistic" I refer to realistic clock diffusion coefficient values.